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Alternative wave-function theory in homogeneous anisotropic media: Spline approximation

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An alternative wave-function theory based on the spline approximation of the undetermined angular-spectrum amplitudes in homogeneous anisotropic media is developed. The scattering of a scalar wave by an anisotropically coated cylinder is considered as an illustrative example. Numerical results for a bistatic radar cross section are presented for both transverse electric and transverse magnetic polarizations to validate our wave-function theory. The present method is applicable to scalar, vector, and tensor fields in homogeneous anisotropic media and any bounded coordinate systems.

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The accurate analysis of waves in anisotropic media is required in mechanics, classical optical electronics, classical acoustics, electromagnetics, and fluid physics. While Refs. [1–4] follow the classical way to expand the undetermined angular-spectrum amplitudes for anisotropic media, this Brief Report takes the spline functions [5] in modern approximation theory to expand the undetermined angular-spectrum amplitudes. The locally compact property of spline functions reduces the range of integration in the coefficients of wave-function series, and removes the oscillation of the integrand in the previous theory [1]. The harmonic factor $\exp(j\omega t)$ is assumed and suppressed throughout.

As shown in Fig. 1, in the region $a \le \rho \le b$, there is a homogeneous anisotropic medium characterized by the following permittivity and permeability tensors:

$$\underline{\varepsilon}=\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & 0\\ \varepsilon_{yx} & \varepsilon_{yy} & 0\\ 0 & 0 & \varepsilon_{zz}\end{bmatrix},\qquad \underline{\mu}=\begin{bmatrix}\mu_{xx} & \mu_{xy} & 0\\ \mu_{yx} & \mu_{yy} & 0\\ 0 & 0 & \mu_{zz}\end{bmatrix}. \tag{1}$$

We will analyze only the case of H (TE) polarization (i.e., $\mathbf{H}=H_z\hat{\mathbf{z}}$); the similar formulation of E (TM) polarization is omitted. The partial differential equation for H polarization is found to be [6]

$$\varepsilon_{xx}\frac{\partial^2 H_z}{\partial x^2}+\varepsilon_{yy}\frac{\partial^2 H_z}{\partial y^2}+(\varepsilon_{yx}+\varepsilon_{xy})\frac{\partial^2 H_z}{\partial x\,\partial y}+\omega^2\mu_{zz}\gamma H_z=0\ , \tag{2}$$

$$\gamma=\varepsilon_{xx}\varepsilon_{yy}-\varepsilon_{xy}\varepsilon_{yx}\ . \tag{3}$$

It can be shown that the angular-spectrum expansion of the field inside a circle in the above anisotropic medium is [6]

$$H_z(\rho,\varphi)=\int_0^{2\pi}d\varphi_k C(\varphi_k)e^{jk(\varphi_k)\rho\cos(\varphi-\varphi_k)}\ , \tag{4}$$

where

$$k(\varphi_k)=\left[\frac{m_z^2}{\varepsilon_+ +\varepsilon_-\cos2\varphi_k+\sigma_+\sin2\varphi_k}\right]^{1/2}, \tag{5a}$$

$$m_z=\omega\sqrt{\mu_{zz}\gamma}\ , \tag{5b}$$

$$\varepsilon_\pm=\tfrac{1}{2}(\varepsilon_{xx}\pm\varepsilon_{yy})\ , \tag{5c}$$

$$\sigma_\pm=\tfrac{1}{2}(\varepsilon_{xy}\pm\varepsilon_{yx})\ . \tag{5d}$$

An alternative way of expanding the undetermined angular-spectrum amplitude is to use B splines [5]. To this end, in the interval $[0,2\pi]$, let

$$0=\varphi_{k0}<\varphi_{k1}<\varphi_{k2}<\cdots<\varphi_{kS}=2\pi\ , \tag{6a}$$

$$\varphi_{ks}=\varphi_{k0}+sh,\quad h=2\pi/S\quad (s=0,\ldots,S)\ . \tag{6b}$$

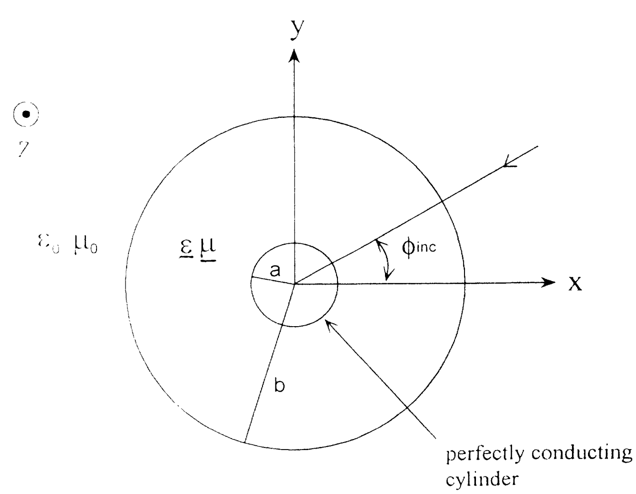

FIG. 1. Geometry of the problem.

Then we have [5]

$$C(\varphi_k)=\sum_{s=0}^{S} a_s B_s(\varphi_k)\ , \qquad (7a)$$

$$B_s(\varphi_k)=B_3(\varphi_k/h-s)\ , \qquad (7b)$$

where

$$B_3(x)=\tfrac{1}{6}\times\begin{cases}(x+2)^3\ , & x\in[-2,-1]\\ (x+2)^3-4(x+1)^3\ , & x\in[-1,0]\\ (2-x)^3-4(1-x)^3\ , & x\in[0,1]\\ (2-x)^3\ , & x\in[1,2]\\ 0, & |x|\geq 2\end{cases} \qquad (7c)$$

is the B spline [5]. Notice that the B spline is locally compact, i.e., $B_3\equiv 0$ if $|x|\geq 2$.

Substituting Eq. (7) into Eq. (4), and expanding the plane-wave factor, we obtain

$$H_z(\rho,\varphi)=\sum_{s=0}^{S} a_s H_{zs}^{(1)}(\rho,\varphi)\ , \qquad (8a)$$

$$H_{zs}^{(1)}(p,\varphi)=\sum_{m=-\infty}^{+\infty} H_{zsm}^{(1)}(\rho)e^{-jm\varphi}\ , \qquad (8b)$$

$$H_{zsm}^{(1)}(\rho)=\int_0^{2\pi} j^{-m}J_m(k(\varphi_k)\rho)B_s(\varphi_k)e^{jm\varphi_k}d\varphi_k\ . \qquad (8c)$$

The compact property of B splines removes the oscillation of the integrand in the previous theory [1] due to the higher-order harmonic functions. Notice that because Bessel functions of other kinds satisfy the same differential equations and the same recursive relations as do Bessel functions of the first kind, we may use Eq. (8) as the general definition of cylindrical wave functions of various kinds for an anisotropic medium if $J_m(k(\varphi_k)\rho)$ is replaced by the corresponding kind.

In the region $a\leq\rho\leq b$, the field is expanded by the newly developed wave functions of the third and fourth kind (or the first and second kind). In the region $\rho\geq b$, the fields are expanded by the well known wave functions for isotropic medium. The boundary conditions lead to a matrix equation. Our results are checked against results based on the exact series solution for a gyrotropic-type dielectric and available from [7], where an integral equation solver was used.

We present in Fig. 2 the bistatic radar cross section (RCS) of the anisotropically coated conducting cylinder for the TE case of $\varphi^{\rm inc}=180°$. Our first calculation is for the RCS of a gyrotropic-type [6] dielectrically coated circular cylinder for the TE case, as shown in Fig. 2(a). The geometry and material parameters in Fig. 2(a) are $k_0b=2$, $k_0a=1$, $\varepsilon_{xx}=4\varepsilon_0$, $\varepsilon_{yy}=4\varepsilon_0$, $\varepsilon_{xy}=-\varepsilon_{yx}=2\varepsilon_0$, $\mu_{zz}=2\mu_0$. For this medium, the exact series solution to the wave equation (2) can be obtained since the $k(\varphi_k)$ is a constant. From Fig. 2(a), it can be seen that our results (solid line) are in excellent agreement with the exact solu-

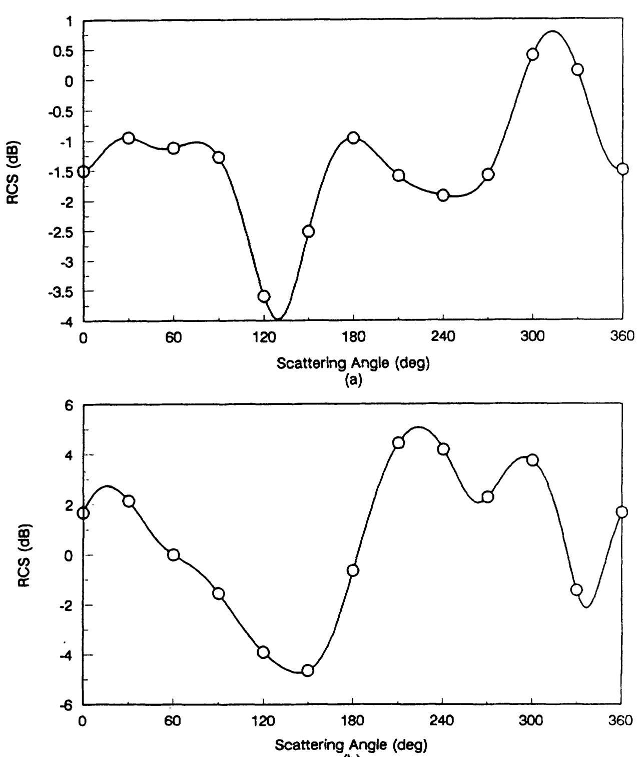

FIG. 2. H polarization, bistatic RCS $(\sigma/\lambda)dB$, $\varphi^{\rm inc}=180°$. (a) $\varepsilon_{xx}=4\varepsilon_0$, $\varepsilon_{yy}=4\varepsilon_0$, $\varepsilon_{xy}=-\varepsilon_{yx}=2\varepsilon_0$, $\mu_{zz}=2\mu_0$, $k_0a=1$, $k_0b=2$; the markers are calculated by the exact series solution. (b) $\varepsilon_{xx}=1.5\varepsilon_0$, $\varepsilon_{yy}=2.5\varepsilon_0$, $\varepsilon_{xy}=-\varepsilon_{yx}=3\varepsilon_0$, $\mu_{zz}=1.5\mu_0$, $k_0a=1$, $k_0b=2$; the markers are the solution taken from Fig. 9(c) of Ref. [7].

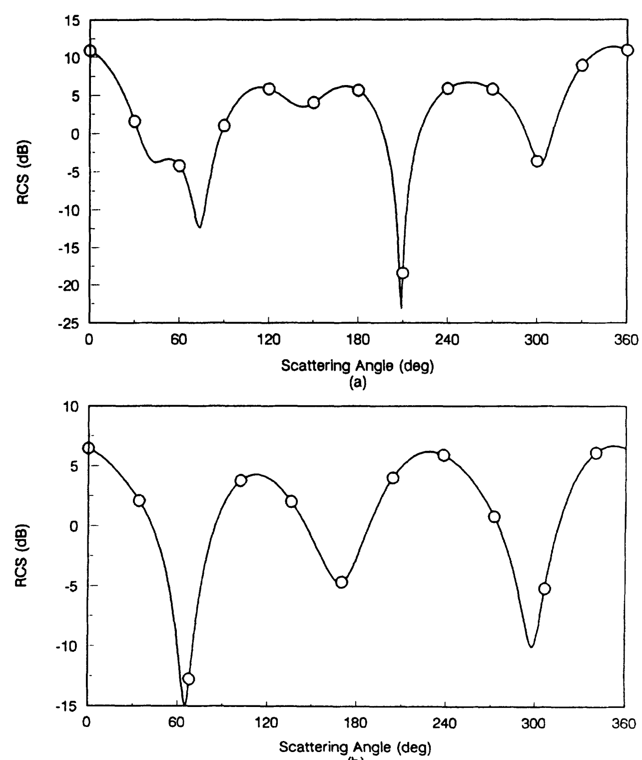

FIG. 3. E polarization, bistatic RCS $(\sigma/\lambda)dB$, $\varphi^{\rm inc}=180°$. (a) $\mu_{xx}=4\mu_0$, $\mu_{yy}=4\mu_0$, $\mu_{xy}=-\mu_{yx}=2\mu_0$, $\varepsilon_{zz}=2\varepsilon_0$, $k_0a=1$, $k_0b=2$; the markers are calculated by the exact series solution. (b) $\mu_{xx}=1.5\mu_0$, $\mu_{yy}=2.5\mu_0$, $\mu_{xy}=-\mu_{yx}=3\mu_0$, $\varepsilon_{zz}=1.5\varepsilon_0$, $k_0a=1$, $k_0b=2$; the markers are the solution taken from Fig. 6(c) of Ref. [7].

tion (markers) since the two curves are indistinguishable. This partly validates the wave-function representations of this Brief Report. For the sake of comparison, the geometry and material parameters in Fig. 2(b) are the same as those of Fig. 9(c) of Ref. [7], i.e., $\varepsilon_{xx}=1.5\varepsilon_0$, $\varepsilon_{yy}=2.5\varepsilon_0$, $\varepsilon_{xy}=-\varepsilon_{yx}=3\varepsilon_0$, $\mu_{zz}=1.5\mu_0$, $k_0a=1$, $k_0b=2$. It should be mentioned that in our notation a and b are two times those in Ref. [7]. Both the results of this Brief Report (solid line) and those of Beker, Umashankar, and Taflove [7] (markers) are displayed in Fig. 2(b). The agreement is excellent.

To illustrate the applicability of the present wave-function solution to the TM case of scattering by an anisotropically coated circular cylinder, the RCS's are calculated for the TM case as shown in Fig. 3. The parameters of Fig. 3 for the TM case are the same as those of Fig. 2 for the TE case. The markers in Fig. 3(a) are calculated by the exact series solutions, the markers in Fig. 3(b) represent the numerical results given in Fig. 6(c) of Ref. [7]. Also, it can be seen from Fig. 3 that the agreement is excellent for the TM case.

All the results are also checked against our previous calculations [8] based on the previous wave-function theory [1–4]. All the numerical results based on the present theory are indistinguishable from the figures based on the previous theory [1,8]. Therefore our wave functions of various kinds are numerically checked. There are many topics in the application of the present wave functions, which are left for future papers.

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